TechScape: The Science, Technology and Education Journal of IIT Jodhpur
Volume 01, Issue 03, August 2021

Superconvergence results for the integral and integro-differential equations and corresponding Eigen value problems by projection methods

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Research Snippets

It is well known that the study of many processes of the natural sciences, engineering and modern world can be reduced to an integral equation and integro-differential initial and boundary value problems to set up, solve, and interpret complicated mathematical models. The problem concerns differential equations, integral equations, and a system of equations derived from conditions on initial or other points, so in most of the cases these equations become too complicated to be solved explicitly. Thus we need to approximate the solution numerically. There has been a notable interest on solving the integral and integro-differential equations of second kind by various numerical techniques. But it will be more profitable to adopt some efficient, convenient method based on piecewise and global polynomial based projection methods such as Galerkin, collocation, multi-Galerkin, multi-collocation methods and their iterated versions to get asymptotic convergence rates for the approximate solutions.

In [1], we considered a class of derivative dependent Fredholm-Hammerstein integral equations i.e., the integral equation, where the nonlinear function inside the integral sign is dependent on derivative and the kernel function is of Green's type. We proposed the piecewise polynomial based Galerkin and iterated Galerkin methods to solve these type of derivative dependent Fredholm-Hammerstein integral equations. We discussed the convergence and error analysis of the proposed methods and also obtained the superconvergence results for iterated Galerkin approximations. Some numerical results are given to illustrate this improvement.

In ([2], [3]), we developed a Jacobi spectral Galerkin method for weakly singular linear and nonlinear Volterra integral equations of the second kind. We applied some variable and function transformations to convert the equation into the new equation, so that the solution of the transformed equation possesses the better regularity results and theory of Jacobi polynomials can be applied in the approximation. The motivation to consider the Jacobi spectral method is that the singularity of the kernel of the Volterra integral equation is incorporated in the weight function to obtain the superconvergence results. Here, we provided the convergence rates of the approximate solution in two cases, when the exact solution is sufficiently smooth and when the exact solution is non-smooth. Theoretical results are verified by numerical illustrations.

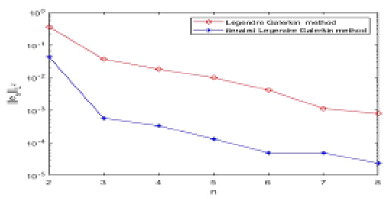
In [4], we discussed the discrete Legendre spectral and iterated discrete Legendre spectral methods for the convergence analysis for Hammerstein type weakly singular nonlinear Fredholm integral equations, with algebraic and logarithmic type kernels. In fact, by choosing the minimal quadrature rule, i.e. the number of quadrature nodes equal to the dimension of the approximating subspace, we obtained the optimal convergence rates in discrete Legendre spectral method in L_2 norm and obtained the optimal convergence rates in iterated discrete Legendre spectral method in L_2 and uniform norm. In [5], we also discussed the discrete Legendre projection methods for a weakly singular compact integral operator to find the error bounds for the approximate eigenfunctions. We showed that eigenfunctions in the iterated discrete Legendre Galerkin method have optimal convergence rates in L_2 and uniform norm. Numerical examples are presented to illustrate the theoretical results.

Recently in [6], we focus on finding the approximate solution for a large class of integro-differential equations of second kind, which is necessary but difficult due to the fact that the differential operator is not bounded in general. Let X= C[-1,1]. Consider the following non-linear Fredholm integro-differential equation of the second kind

$$x^{(n)}(s) - \int_{-1}^{1} k(s,t)\,\phi(t,x(t))\,dt = f(s), \quad -1 \le s \le 1 \cdots\cdots\cdots\cdots\cdots (1)$$

$$x(-1) = 0, x'(-1) = 0, \dots\dots, x^{(n-1)}(-1) = 0,$$

where the kernel k, right hand side function f and ϕ are known functions, x is the unknown function to be determined. It is also known that the quality of the approximate solution of the corresponding operator equations in function space depends essentially on the smoothness of the functions, which define the equation. Here the restriction concern continuity or discontinuity of the Green's kernel, which arises due to the semihomogeneous differential operator. In spite of this difficulty, there has been a notable interest on solving the integro-differential equations of second kind by various numerical techniques. However, there is no literature available for the convergence analysis for higher dimensional nonlinear integro-differential equations. Here, our research work concerns the mathematical analysis of existence, uniqueness, convergence rates of the approximate solution by spectral projection methods for n^th order linear and nonlinear integro-differential initial and boundary value problems with smooth kernel given by (1). We also discuss the significant features of these methods as well as shed some light on advantages of one method over the other (For example, for n=2, with the kernel function k (s ,t)=s(s^2-t), and the function f(s)= 5/4-1/3 s^2 and ϕ(t,x(t))=((x(t)))^2 in equation (1), the numerical error comparison of the proposed methods are given in Fig 1 & Fig 2). We also give the numerical implementations to show that the size of the system of equations that must be solved in multi projection methods remain same as in projection methods. Numerical aspects are also considered to illustrate the theoretical results.



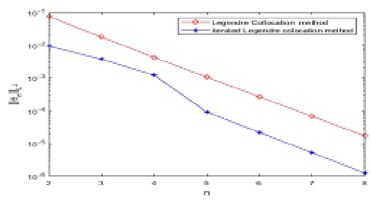


Fig 1. Numerical error comparison between Legendre Galerkin and iterated Legendre Galerkin methods / Legendre collocation and iterated Legendre collocation methods in L^2 norm

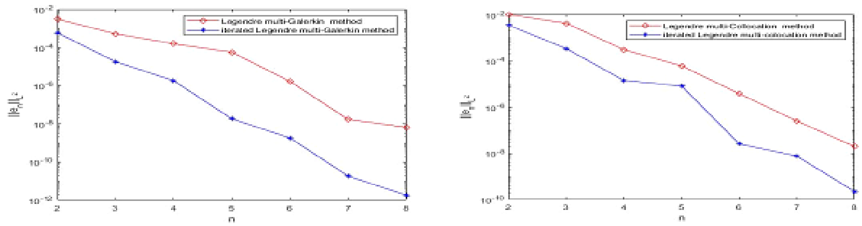


Fig 2. Numerical error comparison between Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods/ Legendre multi-collocation and iterated Legendre multi- collocation methods inL^2 norm

Further Details:-

1. Moumita Mandal, Kapil Kant and Gnaneshwar Nelakanti, Convergence analysis for derivative dependent Fredholm-Hammerstein integral equations with Green's kernel, Journal of Computational and Applied Mathematics 370 (2020), 112599
2. 2. Kapil Kant, Moumita Mandal and Gnaneshwar Nelakanti, Error Analysis of Jacobi spectral Galerkin and multi Galerkin methods for Weakly singular Volterra Integral equations, Mediterranean Journal of Mathematics, 17(1) (2020), 20
3. 3. Kapil Kant, Moumita Mandal and Gnaneshwar Nelakanti, Jacobi Spectral Galerkin Methods for a Class of Nonlinear Weakly Singular Volterra Integral Equations, Advances in Applied Mathematics and Mechanics (2020) (Accepted).
4. 4. Moumita Mandal, Kapil Kant and Gnaneshwar Nelakanti, Discrete Legendre Spectral Methods for Hammerstein type Weakly Singular Nonlinear Fredholm Integral Equations, International Journal of Computer Mathematics (2021) 1-19
5. 5. Moumita Mandal, Kapil Kant and Gnaneshwar Nelakanti, Eigenvalue Problem of a Weakly Singular Compact Integral Operator by discrete Legendre Projection Methods, Journal of Applied Analysis and Computation (2021) (Accepted).
6. 6. Moumita Mandal and Gnaneshwar Nelakant, Approximation solution for a class of non-linear Fredhlom integro-differential equations by projection method (In process).

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