A study of mixed type elliptic problems

Lovelesh Sharma (P20MA209)

Thesis Supervisor: Dr. Tuhina Mukherjee

Department of Mathematics
Indian Institute of Technology Jodhpur

Abstract

The study of elliptic problems involving spectral fractional Laplacian and mixed local-nonlocal operators under mixed boundary conditions opens up rich and intriguing mathematical challenges. This research explores the existence, nonexistence, multiplicity, and regularity of weak solutions for a variety of eigenvalue problems and nonlinear elliptic equations, particularly those with power-type, singular, and critical nonlinearities. The objective is to enhance the understanding of how local and nonlocal effects influence the behavior of solutions in such problems.

The investigation begins with problems driven by the spectral fractional Laplacian under mixed Dirichlet-Neumann boundary conditions. In this setting, the focus is on analyzing singular and critical nonlinearities. By employing variational techniques, energy estimates, and extension methods introduced by Caffarelli and Silvestre, we establish the existence of weak solutions using variational arguments over the Nehari manifold and obtain precise qualitative properties.

Next, we examine an Ambrosetti-Prodi type problem involving the mixed local-nonlocal operator $\mathcal{L} = -\Delta + (-\Delta)^s$ under the Dirichlet boundary conditions. The problem is characterized by the interplay between the parameter λ and critical growth nonlinearities. Using Linking theorems and compactness conditions, we establish the existence of multiple solutions and analyze their dependence on the parameter λ. The study extends classical results for purely local operators to the mixed local and nonlocal setting.

Furthermore, we investigate eigenvalue problems involving mixed operators under mixed boundary conditions. A major part of this work is dedicated to developing a functional framework for handling problems of this type, which serves as a foundation for spectral analysis and bifurcation results. We characterize the first eigenvalue, establish spectral properties based on the topology of the mixed boundary sets, and derive existence results for positive eigenfunctions. The results are then applied to study nonlinear problems, where we prove the existence of solution branches bifurcating from zero and infinity.

We also analyze nonlinear problems involving concave-convex nonlinearities and perturbed singular terms, proving the existence and multiplicity of solutions using the Nehari manifold approach. The regularity of solutions is established through Moser iteration, Stampacchia's method, and Sobolev-type inequalities adapted to the mixed operator setting. Additionally, we derive a strong maximum principle and comparison principles, which are crucial in understanding the qualitative properties of solutions.

Overall, this work aims to contribute meaningfully to the theoretical development of partial differential equations of mixed type. The insights gained here have the potential to impact various scientific fields, such as physics, population dynamics, and models of anomalous diffusion, by advancing our understanding of the mathematical structures underpinning complex systems.

Thesis Publications/Pre-prints

1. T. Mukherjee, P. Pucci, **L. Sharma**: Nonlocal critical exponent singular problems under mixed Dirichlet-Neumann boundary conditions. **Journal of Mathematical Analysis and Applications**, (2023).

2. T. Mukherjee, **L. Sharma**: On critical Ambrosetti–Prodi type problems involving mixed operator. **Journal of Elliptic and Parabolic Equations**, (2024).

3. T. Mukherjee, **L. Sharma**: On elliptic problems with mixed operators and Dirichlet-Neumann boundary conditions. **Nonlinear Differential Equations and Applications NoDEA**, (2025).

4. T. Mukherjee, **L. Sharma**: On singular problems associated with mixed operators under mixed boundary conditions. **Journal of Fixed Point Theory and Applications**, (2025).

5. J. Giacomoni, T. Mukherjee, **L. Sharma**: On an eigenvalue problem associated with mixed operators under mixed boundary conditions. **Discrete and Continuous Dynamical Systems**, (2025).

6. **L. Sharma**: Spectrum of mixed operators under the mixed boundary conditions, **(minor revision submitted)**.

Other Publications/Pre-prints

1. **L. Sharma**: Brezis Nirenberg type results for local non-local problems under mixed boundary conditions. **Communications in Analysis and Mechanic**, (2024).

2. D. Mahanta, T. Mukherjee, A. Sarkar, **L. Sharma**: On the study of (P, Q)-Laplace Choquard equations with critical Trudinger-Moser nonlinearity in H^N, **Analysis and Applications**, (2025).