Projection methods for fractional-order integro-differential equations

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Abstract

Integer-order derivatives and integrals have clear physical and geometric interpretations, but fractional-order integration and differentiation lack such direct understanding. To address this, we first explore the geometric interpretation of fractional derivatives. We establish that the absolute value of a fractional derivative is inversely proportional to the area of a triangle and numerically verify this for some standard functions. Additionally, we discuss the physical interpretation of fractional derivatives in terms of fractional divergence and explore an application of fractional tangent lines in the field of numerical analysis.

Fractional-order integro-differential equations (FIDEs) are widely used in science and engineering to model complex phenomena, but singularities in fractional derivatives pose challenges in finding exact solutions. This thesis develops efficient numerical projection methods for ap proximating solutions of various classes of linear and nonlinear FIDEs with smooth or weakly singular kernel functions. We consider Caputo's fractional derivative as it carry integer-order initial conditions, which have a clear physical interpretation.

We first analyze linear fractional Volterra integro-differential equations (FVIDEs) with smooth kernels and smooth solutions using a piecewise polynomial approximation subspace. To achieve the desired convergence rates, the FVIDEs are transformed into a system of fractional Fredholm integro-differential equations using properties of fractional derivative and suitable variable transformation. Then, we establish general frameworks of the Galerkin and multi Galerkin methods and achieve improved convergence rates in the supremum norm. However, note that the singularity in the fractional derivative operator causes the exact solution to be non-differentiable at the origin. Besides, piecewise polynomial spaces become computationally expensive for higher convergence rates. To address these, we later introduce the shifted Jacobi spectral Galerkin method (SJSGM) and its iterated version for solving linear FVIDEs with non-smooth solutions. The FVIDEs are reformulated into equivalent classical Fredholm integral equations having weak singularities in the kernel functions, with Hölder continuous solution. We achieve improved convergence rates by incorporating the kernel's singularity into the Jacobi weight function and utilizing the solution's Hölder continuity. A smoothing transformation further enhances the regularity of the solution, leading to superconvergence rates.

Next, we establish the convergence results of SJSGM for a more general class of nonlinear weakly singular fractional Volterra-Fredholm integro-differential equations. The equations are transformed into equivalent singular Fredholm integral equations, where the kernels exhibit distinct regularity behaviors. This distinct behavior is effectively handled using Jacobi weight functions, leading to better convergence rates. The Kumar-Sloan method is applied to reduce the computational complexity arising from the nonlinearity. Further, we establish superconvergence rates for the proposed methods using a regularization technique that enhances the smoothness of the solution.

Lastly, we extend the framework of SJSGM, originally developed for a single-form equation, to systems of nonlinear multi-order FVIDEs with smooth or weakly singular kernels having non-smooth solutions, and establish similar superconvergence results using the regularization technique. Numerical experiments rigorously validate the theoretical results, demonstrating the accuracy and effectiveness of the proposed methods.

List of Publications

1. **Ruby**, Mandal, M.:Superconvergent Jacobi spectral Galerkin method for nonlinear fractional Volterra-Fredholm integro-differential equations. **Numerical Algorithms** (2025).

2. **Ruby**, Mandal, M.: Superconvergent results for fractional Volterra integro-differential equations with non-smooth solutions. **Journal of Computational and Applied Mathematics** 458, 116337 (2025).

3. **Ruby**, Mandal, M.: The geometrical and physical interpretation of fractional order derivatives for a general class of functions. **Mathematical Methods in the Applied Sciences** 47, 8400–8420 (2024).

4. **Ruby**, Mandal, M.: Convergence analysis and numerical implementation of projection methods for solving classical and fractional Volterra integro-differential equations. **Mathematics and Computers in Simulation** 225, 889–913 (2023).

5. **Ruby**, Mandal, M.: Optimal and superconvergent rational spectral projection methods for FDEs on unbounded domains. (**Under Review**)

6. **Ruby**, Mandal, M.: A high-accuracy numerical method for nonlinear systems of multi-order fractional Volterra integro-differential equations. **(Submitted)**