Dynamics and Reduced-order Modeling of Coupled Phenomena: A Hamiltonian Approach

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Abstract

Coupled systems arise naturally in physics, engineering, and biology, where the dynamics of one component influence the behavior of the other. Examples include coupled LC circuits, mass-spring systems, pendulums, optical waveguides, resonators, and complex networks. A standard way to analyze such systems is by solving the governing differential equations with suitable initial and boundary conditions. However, analytical solutions are rarely feasible, and purely numerical approaches are often computationally expensive and may not provide much physical insight.

Reduced-order models play a crucial role in simplifying the analysis of coupled systems while retaining the essential dynamics. Among these, Coupled Mode Theory (CMT) has been widely used to describe energy exchange between interacting subsystems, particularly in the case of weak coupling. One of the pioneering physicists, Haus, proposed a phenomenological CMT model in the time domain using a coupled LC circuit as a representative example, which has since been applied to a wide range of systems, including photonics, circuits, and waveguides. However, Haus's model is based on several approximations and is primarily valid for identical systems. Important questions remain unanswered as

- Can we introduce a first principal formulation to the Haus model?
- Can CMT be systematically derived rather than intuitively assumed?
- What is the order of error in existing CMT models?
- How should CMT be extended to nonlinear systems?

In this work, we address these questions using coupled LC circuits as a toy model. First, by introducing the Hamiltonian formulation for the CMT model, we derive an exact model in terms of forward- and backward-propagating waves, and systematically construct an improved CMT model. We then extend the analysis to nonlinear LC circuits containing Josephson junctions, where varying the critical current leads to pitchfork bifurcations in two dimensions: a circle of equilibria emerging after the bifurcation. Motivated by these results, we derive the normal form of pitchfork bifurcations in two dimensions, extend it to the complex plane, and generalize to higher dimensions. In this setting, we also identify a novel bifurcation phenomenon: a circle of equilibria that emerges while the central stability remains unchanged, distinct from the standard pitchfork bifurcation.

Finally, we propose an improved nonlinear CMT model for coupled nonlinear phenomena that achieves significantly better accuracy than the standard model, even under moderately strong coupling. The results establish a mathematical foundation for CMT and open pathways for analyzing more complex systems, such as neural circuits with Josephson junctions and Hamiltonian formulations with losses.

Publications

1. Shubham Garg and Kirankumar R. Hiremath, *Improved reduced order model for study of coupled phenomena,* IOP Journal of Physics A: Mathematical and Theoretical, 57, 415202, 2024 (DOI).
2. Shubham Garg and Kirankumar R. Hiremath, *Bifurcation study of nonlinear LC circuit containing Josephson junction: Comparison of reduced and exact models,* Nonlinear Dynamics (Springer Nature), 113, 15503-15516, 2025 (DOI).
3. Shubham Garg and Kirankumar R. Hiremath, *Nonlinear Coupled Mode Framework for Coupled Systems: From Exact Hamiltonian Models to Improved Reduced Order Formulations,* Chaos: An Interdisciplinary Journal of Nonlinear Science, AIP Publishing, 36, 023119, 2026(DOI).
4. Shubham Garg and Kirankumar R. Hiremath, *Pitchfork bifurcation in higher dimensions and its application to study dynamics of nonlinear resonant cavities,* (Submitted).