Topological Dynamics of Triangular Systems: A Non-autonomous Approach

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Abstract

Dynamical systems are ordered pairs (X, f), where X is a compact metrizable space and $f : X \to X$ is a continuous surjective map on X. Topological dynamics is one of the disciplines of dynamical systems that studies the qualitative characteristics of dynamical systems from a topological perspective. By *triangular map*, we mean a continuous surjective map $T : X \times Y \to X \times Y$ of the form $T(x, y) = (f(x), g(x, y))$. Triangular maps gained attention of several researchers when Kloeden established Sharkovsky's ordering for the triangular maps on the unit square. With the advancement in the knowledge of the dynamics of continuous maps on intervals, triangular maps have become the subject of research as they enable to extend some important characterizations of continuous interval maps. In the late 1900's the problem of classification of triangular maps with zero topological entropy attracted the attention of several researchers, and many interesting characterizations were obtained. Motivated by these advancements, the notion of topological entropy has been further extended and studied within the framework of non-autonomous systems. As one of the components of a triangular system evolves as a non-autonomous system, thus a triangular system can be seen as a conjunction of autonomous and (a family of) non-autonomous systems.

The primary focus of this work is to investigate various dynamical notions for triangular systems and exploring the interplay between dynamics of triangular systems and its non-autonomous component systems. Since chaotic behavior is central to the qualitative study of dynamical systems, we examine several refined notions of transitivity and sensitivity specifically within the context of triangular systems. Additionally, we established necessary and sufficient conditions under which a triangular system exhibits equicontinuity. We also studied the relationship between the non-autonomous systems generated by two "neighboring" points in X and derive conditions under which various forms of mixing and sensitivity in corresponding component systems share similar dynamical characteristics.

We further delve into different forms of shadowing for general triangular systems. In particular, we investigate relation between various notions of shadowing for a triangular system and corresponding shadowing notions for its component systems. We also explore chain transitivity and chain mixing within general triangular systems. We derive conditions under which non-autonomous component systems generated through neighboring points

exhibits similar dynamical behavior.

We also introduce and study different notions of independence for general non-autonomous systems and utilize its framework to investigate minimal triangular systems. As the product system represents a special case when the fiber maps coincide, we extend our study (of independence) to the case when underlying systems are autonomous in nature. We establish conditions under which independence and disjointness can be related for autonomous systems and explore independence to study the dynamical structure of the system. We derive sufficient conditions for a triangular system to be minimal and illustrate the results obtained through explicit examples. We also provide counterexamples to establish the necessity of the condition imposed.

References

[1] F. Balibrea, J. Smítal, M. Štefánková, The three versions of distributional chaos, *Chaos, Solitons and Fractals*, 23 (2005), 1581-1583.

[2] T. Downarowicz, Minimal subsystems of triangular maps of type 2^∞; Conclusion of the Sharkovsky classification program, *Chaos, Solitons & Fractals*, 49 (2013), 61–71.

[3] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, *Mathematical systems theory*, 1(1) (1967), 1-49.

[4] W. Huang, X. Ye, Dynamical systems disjoint from any minimal system, *Transactions of the American Mathematical Society*, 357(2) (2005), 669–694.

[5] A. Iwanik, Independent sets of transitive points, *Banach Center Publications*, 23(1) (1989), 277–282.

[6] P.E. Kloeden, On Sharkovsky's cycle coexistence ordering, *Bulletin of the Australian Mathematical Society*, 20(2) (1979), 171–177.

[7] S. Kolyada, On dynamics of triangular maps of the square, *Ergodic Theory and Dynamical Systems*, 12(4) (1992), 749–768.

[8] S. Kolyada, L. Snoha, Topological entropy of nonautonomous dynamical systems, *Random and Computational Dynamics*, 4(2&3) (1996), 205-233.

[9] J. Smítal, Why it is important to understand dynamics of triangular maps?, *Journal of Difference Equations and Applications*, 14(6) (2008), 597–606.

[10] M. Štefánková, On topological entropy of transitive triangular maps, *Topology and its Applications*, 153(14) (2006), 2673-2679.

List of Publications

[1] **D. Dhawan**, P. Sharma, Various notions of shadowing in triangular system and its component systems, *Topology and its Applications*, 357 (2024).

[2] **D. Dhawan**, P. Sharma, Notions of mixing and sensitivities for triangular map and its non-autonomous components, *Applied General Topology* 26 (2) 2025, 601-615.

[4] **D. Dhawan**, P. Sharma, Independence and related notions for discrete dynamical systems (under review).

[3] **D. Dhawan**, P. Sharma, On dynamics of triangular systems and its component systems (under review).

[5] **D. Dhawan**, P. Sharma, Proximality and Li-Yorke chaos for triangular systems (under preparation).