PDEs Emerging from Moser-Trudinger and Adams Types Inequalities

Deepak Kumar Mahanta (P20MA206)

Thesis Supervisors: Dr. Abhishek Sarkar and Dr. Tuhina Mukherjee

Department of Mathematics, Indian Institute of Technology, Jodhpur

Abstract

This thesis takes advantage of numerous topological tools and variational methods, including the mountain pass theorem, the Ekeland variational principle, the Nehari manifold techniques, the Lusternik-Schnirelmann category theory, etc., to investigate the existence and multiplicity of weak solutions to the nonhomogeneous elliptic and subelliptic PDEs with critical exponential growth in the context of both the Euclidean spaces and the Heisenberg groups. In addition, we are also interested in studying functional inequalities in certain newly generated function spaces in the previously described situations, such as the Adams-type inequalities and the Moser-Trudinger-type inequalities.

In the first work [1], we deal with the existence of nontrivial nonnegative solutions to the following (p,n)-Laplace Schrödinger-Kirchhoff equation:

$$\begin{cases} \mathcal{L}_{p,V}(u)+\mathcal{L}_{n,V}(u)=\lambda_1 h(x)|u|^{q-2}u+\lambda_2\, \dfrac{g(x,u)}{|x|^\beta} \quad \text{in} \quad \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} V(x)\big(|u|^p+|u|^n\big)\,\mathrm{d}x<+\infty,\ u\in W^{1,p}(\mathbb{R}^n)\cap W^{1,n}(\mathbb{R}^n), \end{cases}$$

with

$$\mathcal{L}_{r,V}(u)=M\big(\|u\|^r_{W^{1,r}_V}\big)\big[-\Delta_r u+V(x)|u|^{r-2}u\big]$$

and

$$\|u\|_{W^{1,r}_V}=\left(\int_{\mathbb{R}^n}|\nabla u|^r\,\mathrm{d}x+\int_{\mathbb{R}^n}V(x)|u|^r\,\mathrm{d}x\right)^{\frac{1}{r}} \quad \text{for} \quad r\in\{p,n\},$$

where $n\geq 2$, $\beta\in[0,n)$, $1<p,\ q<n<+\infty$, λ_1 and λ_2 are two positive parameters, $h:\mathbb{R}^n\to(0,+\infty)$ is a weight function in $L^\eta(\mathbb{R}^n)$ with $\eta=\frac{n}{n-q}$, $M:[0,+\infty)\to[0,+\infty)$ is a Kirchhoff function, the scalar potential $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function and $\Delta_m u=\operatorname{div}(|\nabla u|^{m-2}\nabla u)$ with $m>1$ is the m-Laplace operator. The nonlinear term $g:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ has critical exponential growth at infinity. The main features of this problem are the (p,n)- growth of the elliptic operators, the double lack of compactness and the fact that the Kirchhoff function is of degenerate type, that is, it could vanish at zero. To establish the existence results, we use the mountain pass theorem, the Ekeland variational principle, the singular Moser-Trudinger inequality and a completely new Brézis-Lieb type lemma for singular exponential nonlinearity.

In the second work [2], by employing variational methods, the doubly weighted Hardy-Littlewood-Sobolev inequality, the Moser-Trudinger inequality, we deal with the existence of positive ground state solutions to a class of (p,n)-Laplace Schrödinger equations with Stein-Weiss reaction under critical exponential growth in the sense of the Moser-Trudinger in the whole $\mathbb{R}^n$ as follows:

$$\mathcal{L}_{p,V}(u)+\mathcal{L}_{n,V}(u)=\left(\int_{\mathbb{R}^n}\frac{F(y,u)}{|x-y|^\mu|y|^\beta}\,\mathrm{d}y\right)\frac{f(x,u)}{|x|^\beta} \quad \text{in } \mathbb{R}^n,$$

with

$$\mathcal{L}_{m,V}(u)=-\Delta_m u+V(x)|u|^{m-2}u \quad \text{for} \quad m\in\{p,n\},$$

where $1<p<n$ with $n\geq 2$, $\beta>0$, $0<\mu<n$, $0<2\beta+\mu<n$, $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function. The nonlinearity $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ has critical exponential growth at infinity.

In the third work [3], we focus on the study of the existence, multiplicity and concentration behavior of ground states as well as the qualitative aspects of positive solutions to a (p,n)-Laplace Schrödinger equation with logarithmic nonlinearity and critical exponential nonlinearity as follows:

$$\begin{cases} \mathcal{L}_{p_\varepsilon}(u) + \mathcal{L}_{n_\varepsilon}(u) = |u|^{n-2}u\log|u|^n + f(u) \quad \text{in} \quad \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} V(x)\big(|u|^p + |u|^n\big)\,\mathrm{d}x < +\infty, \quad u \in W^{1,p}(\mathbb{R}^n)\cap W^{1,n}(\mathbb{R}^n), \end{cases}$$

where

$$\mathcal{L}_{t_\varepsilon}(u) = -\varepsilon^t\Delta_t u + V(x)|u|^{t-2}u \quad \text{for } t\in\{p,n\}$$

with $n\geq 2$, $1<p<n$ and ε is a very small positive parameter. The scalar potential $V\colon\mathbb{R}^n\to\mathbb{R}$ is a continuous function and the nonlinearity $f\colon\mathbb{R}\to\mathbb{R}$ has critical exponential growth at infinity. Through the use of smooth variational methods, Nehari manifold techniques, penalization techniques and the application of the Lusternik-Schnirelmann category theory, we establish a connection between the number of positive solutions and the topological properties of a set in which the potential function achieves its minimum values.

In the fourth work [4], we establish a sharp version of Adams type inequality in a suitable higher order function space with singular weight in $\mathbb{R}^n$. In addition, we also provide the proof of a concentration-compactness principle inspired by P.L. Lions as an improvement of the Adams inequality. Moreover, we shall establish that critical and subcritical sharp singular Adams-type inequalities are surprisingly equivalent. More precisely, we discuss about the asymptotic behavior of the lower and upper bounds of subcritical sharp singular Adams-type inequality and establish a relation between the suprema of such types of critical and subcritical sharp inequalities. Further, as an application of these results, by employing the mountain pass theorem, we study the existence of nontrivial solutions to a $(p,\frac{n}{2})$-biharmonic equation with singular exponential growth in $\mathbb{R}^n$ as follows:

$$\Delta_p^2 u + \Delta_{\frac{n}{2}}^2 u = \frac{g(x,u)}{|x|^\gamma} \quad \text{in} \quad \mathbb{R}^n,$$

with $n\geq 4$, $1<p<\frac{n}{2}$, $\gamma\in(0,n)$ and $\Delta_t^2 u = \Delta(|\Delta u|^{t-2}\Delta u)$ is a fourth order operator, which is known as the standard t-biharmonic operator for any $t>1$. The nonlinearity $g:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and has critical exponential growth at infinity.

In the last work [5], we first establish a sharp singular Moser-Trudinger type inequality in a completely new function space and then, inspired by P.L. Lions, we also study a sharp singular concentration-compactness principle as an improvement of this Moser-Trudinger inequality in the Heisenberg group context. Moreover, we also discuss about the equivalence of the sharp critical and subcritical singular Moser-Trudinger type inequalities. Specifically, we investigate the asymptotic behavior of the lower and upper bounds of the sharp subcritical singular Moser-Trudinger type inequality and derive a connection between the suprema of such types of the sharp critical and subcritical inequalities. In addition, due to the applications of the above results, by invoking the mountain pass theorem, we study the existence of positive ground state solutions to a zero mass (p,Q)-Laplace subelliptic equations involving singular exponential nonlinearity in $\mathbb{H}^n$ as follows:

$$-\Delta_{H,p}u - \Delta_{H,Q}u = \frac{f(\xi,u)}{r(\xi)^\vartheta} \quad \text{in} \quad \mathbb{H}^n,$$

with $1<p<Q$, $\vartheta\in(0,Q)$ and $Q=2n+2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$. The operator $\Delta_{H,t}u = \operatorname{div}_H(|D_H u|_H^{t-2}D_H u)$ with $t>1$ is the standard t-Kohn–Spencer Laplace (or horizontal t-Laplace) operator. The function $r(\cdot)$ is called the Korányi norm in $\mathbb{H}^n$. The nonlinear term $f:\mathbb{H}^n\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, which satisfies critical exponential growth at infinity.

Publications/Pre-prints from Thesis

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Additional Publications/Pre-prints

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