Qualitative Analysis of Non-Autonomous Dynamical Systems

Sushmita Yadav (P20MA010)
Supervisor: Prof. Puneet Sharma

Department of Mathematics, IIT Jodhpur

The modern theory of dynamical systems traces its origins to the late nineteenth century with the pioneering work of Henri Poincaré, who explored the dynamics of the three-body problem in celestial mechanics. Traditionally, most studies have focused on autonomous systems, where the governing rules remain invariant over time. However, many real-world processes are inherently non-autonomous, with dynamics that evolve under time-dependent rules. To more accurately model these processes, it is essential to consider systems with time-varying behavior.

Qualitative analysis of non-autonomous dynamical systems (NADS) was first studied by Kolyada and Snoha [KS$^+$96] in 1996. In [KS$^+$96], the authors investigate the topological entropy of NADS. Later in [KST04], the authors discuss minimality conditions for NADS. In [BO12], authors investigate dynamical notions such as Li-Yorke chaos, topological weak mixing, and topological entropy for a general non-autonomous dynamical system. These contributions have attracted many researchers in the qualitative analysis of non-autonomous dynamical systems, for instance, see [Dvo12, SR18b, LC17, Šte15, MMAS18, SR18a]. Although progress has been made in studying discrete-time non-autonomous systems, numerous fundamental questions remain open in this setting.

In this thesis work, we present a comprehensive study of the qualitative analysis of non-autonomous discrete systems, with a focus on their asymptotic behavior under various dynamical properties. We begin by investigating some dynamical notions such as minimality, almost periodicity, and equicontinuity for general non-autonomous systems on compact metric spaces as a phase space. Further, we explore various notions of proximality, such as syndetic, regional, and regionally syndetic proximality. We provide a sufficient criterion for proximal and syndetically proximal relations to coincide in a non-autonomous system. We also discuss various proximal notions in product spaces.

In the setting of topological flows, the notion of the enveloping semigroup plays a central role in understanding long-term dynamical behavior [Ell60, Ell93, EN89]. Motivated by this, we introduce an analogous notion for non-autonomous systems, which we call an enveloping cover, though it does not inherit a semigroup structure in the non-autonomous setting. Before delving into the dynamics of non-autonomous sys-

tems via this construction, we first examine the classical case: we explore the enveloping semigroup of a topological flow and study the induced flow it supports. We then relate various dynamical properties of the original flow to those of the induced flow on its enveloping semigroup. With this foundation, we return to the non-autonomous setting and investigate the structure of the enveloping cover. In particular, we establish the equivalence of equicontinuity and distality for non-autonomous systems on the real line using the enveloping cover, and further demonstrate how several topological properties of the enveloping cover reflect the dynamics of the underlying non-autonomous system.

References

[BO12] Francisco Balibrea and Piotr Oprocha. Weak mixing and chaos in nonautonomous discrete systems. *Applied Mathematics Letters*, 25(8):1135–1141, 2012.

[Dvo12] J Dvořáková. Chaos in nonautonomous discrete dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, 17(12):4649–4652, 2012.

[Ell60] Robert Ellis. A semigroup associated with a transformation group. *Transactions of the American Mathematical Society*, 94(2):272–281, 1960.

[Ell93] Robert Ellis. The enveloping semigroup of projective flows. *Ergodic Theory and Dynamical Systems*, 13(4):635–660, 1993.

[EN89] Robert Ellis and Mahesh Nerurkar. Weakly almost periodic flows. *Transactions of the American Mathematical Society*, 313(1):103–119, 1989.

[KS+96] Sergiĭ Kolyada, Lubomir Snoha, et al. Topological entropy of nonautonomous dynamical systems. *Random and computational dynamics*, 4(2):205, 1996.

[KST04] S Kolyada, L' Snoha, and S Trofimchuk. On minimality of nonautonomous dynamical systems. *Nonlinear Oscillations*, 7(1):83–89, 2004.

[LC17] Tianxiu Lu and Guanrong Chen. Proximal and syndetical properties in nonautonomous discrete systems. *Journal of Applied Analysis and Computation*, 7(1):92–101, 2017.

[MMAS18] Alejandro Miralles, Marina Murillo-Arcila, and Manuel Sanchis. Sensitive dependence for nonautonomous discrete dynamical systems. *Journal of Mathematical Analysis and Applications*, 463(1):268–275, 2018.

[SR18a] Puneet Sharma and Manish Raghav. Dynamics of nonautonomous discrete dynamical systems. *Topology Proceedings*, 52:45–59, 2018.

[SR18b] Puneet Sharma and Manish Raghav. On dynamics generated by a uniformly convergent sequence of maps. *Topology and its Applications*, 247:81–90, 2018.

[Šte15] Marta Štefánková. Inheriting of chaos in uniformly convergent nonautonomous dynamical systems on the interval. *Discrete and Continuous Dynamical Systems*, 36(6):3435–3443, 2015.

List of Publications/Pre-prints from the Thesis

1. **S. Yadav** and P. Sharma, "Dynamical Behavior of a General Non-autonomous Dynamical System", *Applied General Topology*, 26 (2025) (1), 203-219.
2. **S. Yadav** and P. Sharma, "Equivalence of Equicontinuity and Distality for Real Non-Autonomous Systems", *Filomat*, 39 (2025) (14), 4907-4912.
3. **S. Yadav** and P. Sharma, "Enveloping Cover of Non-Autonomous Systems", (Under Revision).
4. **S. Yadav** and P. Sharma, "Dynamics of Enveloping Semigroup of Flows", (Under Review).
5. **S. Yadav** and P. Sharma, "Various Notions of Proximality of Non-Autonomous Dynamical Systems", (Under Review).